Unsteady Motion of Airfoils with Boundary-Layer Separation

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It is shown that the condition that determines circulation about an airfoil with boundary layers is identical with the usual inviscid-flow condition based on conservation of total circulation and the Kutta-Joukowski condition, in both steady and unsteady flow. This implies interesting relationships between the viscous and inviscid models, namely, between boundary-layer vorticity and bound-vortex strength, viscous-wake vorticity and free-vortex strength, and vortex/vorticity fluxes, in both steady and unsteady flow. The unsteady aerodynamics of airfoils with rounded trailing edges is discussed in this light, and it is concluded that a dual model is needed, involving a boundary-layer calculation over a smooth body to determine circulation, and a vortex-sheet model to determine the perturbed potential flowfield needed in this calculation, as well as forces and moments on the airfoil.

Nomenclature

A,B	= points of boundary-layer separation on top and bottom surfaces of airfoil
L	= lift
$l.e.$	= leading edge
M_{te}	= pitching moment about trailing edge, positive nose-down
p	= pressure
p_{bottom}, p_{top}	= pressures on bottom and top surfaces of the bound-vortex sheet
Re	= Reynolds number based on characteristic dimension of airfoil
t	= time
x,y	= streamwise and cross-stream coordinates; also boundary-layer coordinates
u,v	= velocity components in x, y directions, relative to fixed frame of reference
u'	= perturbation defined by $u=U+u'$
u_{rel}	= same as u but measured relative to a moving separation point; $u_{\text{rel}}=u-u_{\text{sep}}$
u_{sep}	= streamwise velocity of motion of the separation phenomenon, relative to the fixed frame
u_A, u_B	= values of u, at A and B respectively
u_I	= value of u at the outside edge of the boundary layer
U	= stream speed
γ	= strength of vortex sheet, positive clockwise
Γ	= circulation about airfoil plus boundary layers, positive clockwise
δ	= boundary-layer thickness
ϵ	= order of magnitude of incidence, thickness, camber of thin airfoil and of its unsteady motion
ζ	= vorticity, positive clockwise
ρ	= mass density of fluid
ϕ	= velocity potential
γ_{eff}	= effective strength of vortex sheet, defined by Eq. (6)

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Introduction

ONE of the most profound and interesting observations of the early-twentieth-century fluid mechanicists was the recognition of the intimate relationship between the boundary layer and circulation. Every student of modern aerodynamics is, or should be, familiar with the explanation of the well-known trailing-edge condition of Kutta and Joukowski in terms of boundary-layer separation, vortex shedding, and conservation of total circulation.

The extension of these concepts to flows around bodies that do not have sharp trailing edges is probably not as well known but was carried out convincingly by Howarth,[1] who showed that the counterpart of the Kutta-Joukowski condition, for steady flow, is that equal and opposite fluxes of vorticity occur at the two (upper and lower-surface) points of boundary-layer separation, where the wake originates. Using this result, he calculated the circulation and lift-per-unit-length of an infinite cylinder of elliptic cross section as functions of its angle of attack. He first assumed laminar boundary layers and laminar separation and then made the analogous calculation with turbulent separation. This case is more involved than the purely laminar, of course, because the phenomenon of transition must be modeled.

In 1956, the present author[2] reviewed this subject, among others, and considered the extension of Howarth's condition to unsteady flow. The conclusion was that the total vorticity flux at separation should not vanish but should be related to the instantaneous rate of change of circulation. The argument is not lengthy and is repeated here.

Let A and B denote the boundary-layer separation points on upper and lower surfaces, as in Fig. 1. The vorticity flux at such a point, in the boundary-layer approximation and for a surface of negligible curvature, is

$$\int_0^\delta \zeta u_{\text{rel}}\,dy = \int_0^\delta (u-u_{\text{sep}})\,\frac{\partial u}{\partial y}\,dy \tag{1}$$

where δ denotes the local boundary-layer thickness, u the velocity component parallel to the wall and ζ the vorticity, defined as positive in the clockwise direction, unusual as that may be. If u_{sep} denotes the velocity of motion of the separation phenomenon along the wall, u_{rel} can be written as $u-u_{\text{sep}}$ as indicated.

Integrating, we have

$$\int_0^\delta \zeta u_{\text{rel}}\,dy = \tfrac{1}{2}u_I^2 - u_{\text{sep}}u_I \tag{2}$$

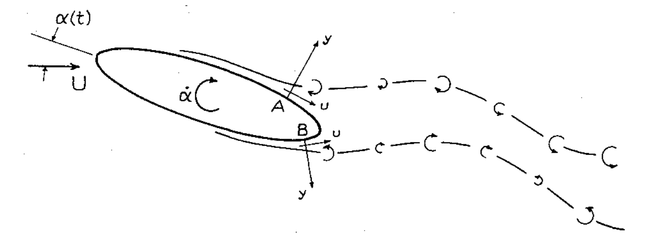

Fig. 1 Sketch showing airfoil in unsteady motion, with vortical wakes produced at boundary-layer separation points A and B.

where u_l is the value of u at the outside edge of the boundary layer. The generalized Howarth criterion is, therefore,

$$[\tfrac{1}{2}u_l^2 - u_{\text{sep}}u_l]_B^A = -\mathrm{d}\Gamma/\mathrm{d}t \tag{3}$$

where $[\quad]_B^A$ denotes the difference between the values at A and B of the expression in the brackets, and Γ denotes the instantaneous circulation about the airfoil. Clearly, in steady flow, both u_{sep} and $\mathrm{d}\Gamma/\mathrm{d}t$ are zero, and Howarth's criterion applies.

The author's attention was drawn back to this subject recently by unpublished work of J.C. Williams III and W.D. Johnson of North Carolina State University. These investigators have undertaken to work out a first-order theory of oscillating airfoils, using the circulation criterion given previously and in Ref. 2, and avoiding the special restriction made by Moore,[3] who treated only small oscillations about the point of maximum circulation.

In this paper, we consider first the classical case of airfoils with sharp trailing edges and show how the preceding circulation criterion is related to the Kutta-Joukowski condition. We then proceed to consideration of flow over airfoils with rounded trailing edges, which is the case considered by Williams and Johnson.

Airfoils with Sharp Trailing Edges

The classical case represented by traditional thin-airfoil theory (see, for example, Ref. 4 and the literature cited there) is that of an airfoil of small thickness, camber, and incidence, having thin boundary layers attached all the way to the trailing edge. The points A and B, where the wake begins, are then both at the trailing edge, and u_{sep} is zero at both A and B. The familiar theory is first-order; that is, quantities of order ϵ are retained and higher-order terms discarded, where $O(\epsilon)$ denotes the order of magnitude of thickness ratio, camber, incidence, and amplitude of oscillation. The velocity components u, as in Eq. (3), for example, are expressed as perturbations of the undisturbed stream speed U; viz., $u = U + u'$, and u' is a quantity of order ϵ.

Then, Eq. (3) becomes simply (to order ϵ)

$$U(u_A' - u_B') = -\mathrm{d}\Gamma/\mathrm{d}t \tag{4}$$

The airfoil and its wake then are represented as vortex sheets, bound and free, respectively, of strength γ, say, and Eq. (4) becomes a statement of the vortex strength at the trailing edge:

$$U\gamma_{te} = -\mathrm{d}\Gamma/\mathrm{d}t \tag{5}$$

Let us recapitulate. The preceding argument, which led us to Eqs. (3-5), was one concerning total vorticity flux at the termination of the boundary layers and conservation of total circulation. In familiar, "inviscid," unsteady thin-airfoil theory, Eq. (5) usually is written on the basis of a somewhat different argument: rate of vortex shedding at the trailing edge is related to rate of change of circulation. We see here a rather surprising identity between vorticity production in the boundary layers and vortex shedding from a time-varying vortex sheet, and again between the net vorticity of the viscous wake and the vortex strength of the free shed-vortex sheet.

There is also an identity between all this and the well-known Kutta-Joukowski condition, although this may not be recognized immediately. The condition usually is presented as the statement that the pressures above and below the trailing edge must be equal. In the absence of the wake of shed vortices, this means that the vortex strength at the trailing edge, γ_{te}, must vanish; but it is easily verified that, in the presence of the wake sheet, it means instead that the vortex strengths of airfoil and wake must be equal at the trailing edge. If they are unequal, at any instant, infinitely large velocities occur at the trailing edge. The strength γ_{te} does not vanish; nevertheless the pressures above and below are equal, by virtue of the contribututions $\rho\partial\phi/\partial t$ there, where ϕ denotes velocity potential. (The difference between $\partial\phi/\partial t$ at top and bottom at the trailing edge is $\mathrm{d}\Gamma/\mathrm{d}t$, as pointed out in Ref. 2.)

We are not, of course, proposing that the vortices that make up the bound-vortex sheet that replaces the airfoil are simply a model of the vortical layers (boundary layers) at the airfoil, although it is certainly true that the total vortex strength (circulation per unit chordwise length) of these layers is exactly the same as $\gamma(x, t)$. There would seem to be some conceptual diffuculties in this identification, such as the fact that normal forces are exerted on the vortex sheet, whereas the boundary layers themselves sustain no normal forces, and that cases can be imagined which have no boundary layers (because the fluid is inviscid or the layer is removed, etc.), and bound vortices are nevertheless appropriate. Nevertheless, what we do say is that the bound-vortex sheet of strength $\gamma(x, t)$ does represent the airfoil plus its boundary layers, and that the shed-vortex wake behind the airfoil does represent simply the vortical viscous wake that forms at the termination of the boundary layers. All of this serves to remind us that the inviscid-fluid model must represent the limiting case of vanishingly small viscosity and not the flow of a truly inviscid fluid.

Thus, the viscous (boundary-layer) and inviscid models of an unsteady airfoil are identical; in both there is a continuous flux of vorticity from the trailing edge into the wake, and there is no discontinuity in vortex strength at the trailing edge.

The circulation Γ is the integral of the bound-vortex strength

$$\Gamma(t) = \int_{le}^{te} \gamma(x,t)\,\mathrm{d}x$$

and therefore is the circulation about airfoil plus boundary layers. The trailing edge is just the chordwise station where the vortex distribution becomes "free" insteady of "bound," because there is no force on the wake.

In this context, the most interesting case is, perhaps, that of steady flow, where again there is continuous flux of vorticity from the boundary layers into the wake, with no discontinuity. What is unique to steady flow is that here the fluxes are equal and opposite above and below the trailing edge, and the vortex strength is zero both on the airfoil at the trailing edge and in the wake.

All that remains to be said about the present class of flow problems is that both the bound- and free-vortex sheets can be replaced by plane sheets lying in the stream direction, and other simplifying approximations can be made, when a theory correct only to $O(\epsilon)$ is constructed. The effects of the displacements of the vortices from this plane, as well as details of their convection velocities, are of higher order, $o(\epsilon)$, so far as conditions on the airfoil are concerned.

Airfoils with Rounded Trailing Edges

In general, for a cylinder with rounded trailing edge, the configuration of boundary layers, separation, and wake must be something like that sketched in Fig. 1. There are separated, vortical layers downstream of separation at both upper and lower surfaces near the trailing edge, forming a viscous wake of nonzero total vorticity behind the trailing edge.

Suppose that an unsteady thin-airfoil theory, analogous to what just has been discussed, is required: again let $O(\epsilon)$ denote the order of magnitude of angle of attack, amplitude of oscillations, thickness ratio, and camber; then aerodynamic quantities such as lift and circulation and their fluctuations are $O(\epsilon)$, and a theory capable of predicting them to this order is wanted.

Now, the circulation criterion, Eq. (3), involves quantities of $O(\epsilon)$, as it should. But, to find the separation points A and B in this kind of flow, and thus to evaluated the terms of Eq. (3), requires a boundary-layer calculation. Consequently, the order of magnitude $Re^{-1/2}$ (or δ) necessarily is introduced. Boundary-layer theory involves quantities of this order, but the important function of this theory in the present application (and this is not unusual) is to give us results that are $O(\delta^0)$.

At this point, it is well to consider both small quantities, ϵ and δ, together. Thus, the required thin-airfoil theory should be correct to $O(\epsilon\delta^0)$, which is, of course, $O(\epsilon)$. It would surely be possible to construct a theory correct to $O(\epsilon\delta)$, for boundary-layer theory is correct to this order, and its information never is exploited completely until all of the results are extracted to this order of accuracy. But this would seem to involve correcting the potential flow for boundary-layer displacements effects, and similar complications. Let us assume, instead, that high-Re flow is being considered and only $O(\epsilon\delta^0)$ is required. Now, for a theory correct to $O(\epsilon\delta^0)$, it is necessary to provide, for the boundary-layer calculation, a potential flow correct to this order. This is provided by extension of the familiar first-order techniques of airfoil theory described previously, which treat the airfoil as a plane vortex sheet attached to a plane vortex wake. The principle behind this simple, familiar model is clearly one of linear superposition: the effects of steady features such as thickness and mean incidence are linearly additive to the effects of airfoil motions, of fluctuations of the points of separation and circulation, and of induced effect of the vortical wake. Furthermore, effects of displacement of the wake from a streamwise plane and of departures of its vortices from translation at speed U make only higher-order perturbations at the airfoil. In the present application, this is true of the vertical displacement and motion of the vortical layers emanating from the upper and lower surfaces. The two-layered wake of Fig. 1, whose thickness is of the same order as the thickness of the cylinder, $O(\epsilon)$, is to be modeled once again simply as a flat sheet whose vortex-strength distribution is the total (net) of the upper and lower shear layers. The drift speed of these layers also is approximated correctly by the undisturbed stream speed U.

But boundary-layer theory is nonlinear, and therefore these principles of linear superposition cannot be used to determine the points of boundary-layer separation and the resulting circulation. The potential flow, correct to $O(\epsilon\delta^0)$, provided for the boundary-layer calculation must include all effects of this order, due to camber, thickness, incidence, instantaneous motion, and wake induction. Furthermore, familiar thin-airfoil theory is nonuniformly valid near the leading edge, and this singularity must complicate the boundary-layer calculation seriously. There are techniques for rendering the results of thin-airfoil theory uniformly valid, such as Goldstein's[5] and Lighthill's[6], but it may be simpler in many cases to resort to "exact" potential-flow calculations (even though the perturbed flow is described only approximately), thus providing more accuracy than needed but avoiding the singularity. This is true for an airfoil of elliptic profile, for example, since it can be transformed easily conformally into a circle. But, in any case, the representation of the thin cylinder as a flat vortex sheet suffices for the calculation of pressure, lift, moment, etc. after the circulation has been established.

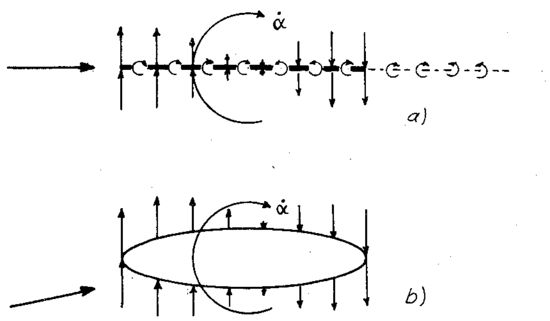

Fig. 2 Dual model for unsteady thin airfoil with rounded trailing edge: a) bound and free vortex sheets; b) model for boundary-layer calculation.

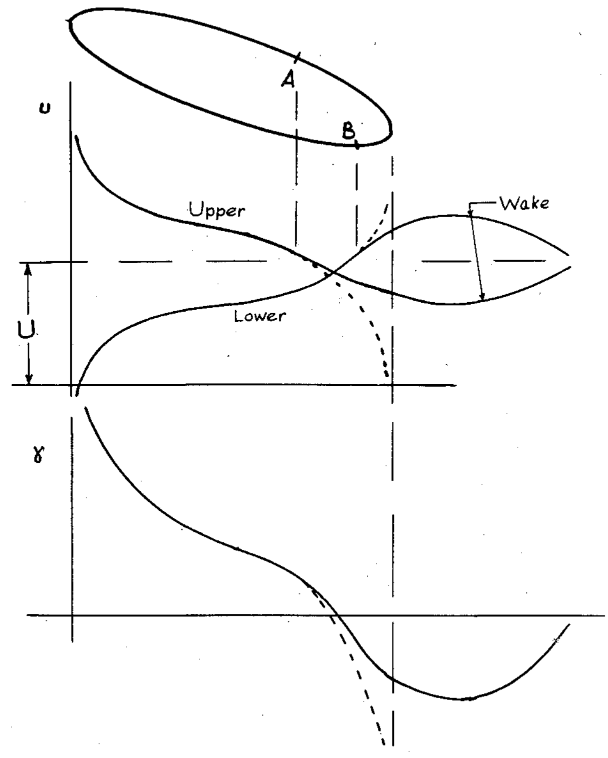

Fig. 3 Sketch showing (above) velocity component u at upper and lower surfaces of vortex sheets, and (below) corresponding vortex strength γ.

What we arrive at is a dual model, as suggested by Fig. 2. Lift, moment, load distribution, etc. are to be calculated as in thin-airfoil theory (Fig. 2a), but circulation is to be determined with the air of a boundary-layer calculation that requires a more complete picture of the airfoil's contour and its stagnation point, namely, a picture that is accurate to $O(\epsilon)$ (Fig. 2b). The two models are not independent; the flowfield disturbances calculated in Fig. 2a are carried over into Fig. 2b, and the circulation determined by Eq. (3) in Fig. 2b is carried back to Fig. 2a, determining the strengths of both bound- and free-vortex sheets.

Conditions Downstream of Separation

The results of the process just described, however, will be a bound-vortex distribution that is singular at the trailing edge (as well as the leading edge), since the circulation is not the Kutta-Joukowski value, and only that value removes the trailing-edge singularity. Presumably, one could accept the trailing-edge singularity, as one does the leading-edge singularity, if it could be considered a reasonable approximation to an actual peaked, pressure/load distribution, as at the leading edge. But there is no actual flow around the trailing edge (cf Fig. 1), and so the singularity is quite inappropriate in a model representing the airfoil plus viscous layers. In fact, our model pictures the viscous wake as originating at points A and B; the vortex strength downstream of these points therefore is determined by the time-history of separation: it is known at any time t if the calculation has been carried out for earlier times.

This situation may be clarified by Fig. 3, where the instantaneous upper- and lower-surface potential-flow velocity perturbations u' are sketched, together with the corresponding γ's. The dashed lines denote the results of superimposing circulation upon the perturbations due to camber, thickness, instantaneous motion, and wake induction. The solid lines represent the more acceptable model just described; viz., the vorticity distributions downstream of separation points A and B are taken from the previous history of separation. Clearly, there are differences in circulation, pressure, load, and lift between these two approximations, and the solid curves should be used to define these quantities.

This means, to be exact, that the potential flow about the cylinder really was not reproduced correctly. But it was used only to determine boundary-layer separation through Eq. (3). The improvements we are suggesting for the regions downstream of separation must have relatively small effects on conditions upstream of separation and therefore on the locations of separation.

To be sure, the refinement suggested here was not made by Howarth,[1] whose calculations for steady flow are subject to the same criticism as made previously. Thus, according to our present argument, Howarth's values of circulation must be somewhat in error: probably too large.

We know that there are flows in which the thickness of the separated layer becomes very large, and cases where upper-surface separation occurs well upstream, near the leading edge. It is believed, on the other hand, that there are other regimes of flow where separated layers remain relatively thin. The theory outlined here is intended for these latter regimes, where separation occurs upstream of the trailing edge and excursions of the separation points have important effects, and yet the separated-layer errors identified here remain unimportant. Otherwise, a more accurate model of separated flow (perhaps empirical) will be required.

Calculation of Forces

Finally, a word should be added about the calculation of lift and the load distribution on the airfoil. The pressure on the airfoil, of course, must be calculated by the "unsteady Bernoulli equation" with the term $\rho\partial\phi/\partial t$, as has been mentioned already. The formulas used in conventional unsteady thin-airfoil theory[4] for the load distribution (pressure difference between bottom and top surfaces) are based upon this equation. The load is not equal to $\rho U\gamma$ in unsteady flow, nor is the lift equal to $\rho U\Gamma$. For example, in Ref. 7 the load is given as

$$p_{\text{bottom}} - p_{\text{top}} = \rho U \gamma_{\text{eff}}(x,t) \tag{6}$$

for points on the bound-vortex sheet (airfoil), where

$$\gamma_{\text{eff}}(x,t) = \gamma(x,t) + \frac{1}{U}\frac{\partial}{\partial t}\int_{le}^{x} \gamma(x,t)\,\mathrm{d}x \tag{7}$$

(The integral is carried from the leading edge back to x).

But there is a questionable point in the use of these formulas in cases where separation occurs forward of the trailing edge. As has been pointed out, the bound-vortex sheet actually represents the airfoil and its viscous layers. According to boundary-layer theory, the pressure on this sheet is equal to the pressure on the airfoil itself, to order $\epsilon\delta^0$ (or better), but boundary-layer theory does not hold at and downstream of boundary-layer separation. If the separated layer becomes very thick, the pressure calculated for the bound-vortex sheet may differ from that at the body's surface.

Nevertheless, the formulas deduced[4] for the total lift and moment on the body, in terms of the vortex-strength distributions, still must be valid, for they were derived from considerations of vertical-momentum and moment-of-momentum production, and the only exterior force and moment that act on the system (airfoil plus vortical layers) are the lift and moment on the airfoil. Stated somewhat differently, we calculate in Eqs. (6) and (7) the loading on airfoil plus viscous layers. But no net force or moment is applied to the viscous layers, even though there are eddies and pressure variations within such layers. So the total force and moment calculated by integration of Eq. (7) are both the force and moment on the ensemble and on the solid body itself.

For example, the following formula for lift is obtained:

$$L(t) = \rho U\Gamma(t) - \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{le}^{te} \gamma(x,t)x\,\mathrm{d}x \tag{8}$$

where x is measured (positive downstream) from the trailing edge [i.e., all values of x in the integral of Eq. (8) are negative]. The analogous formula for the pitching moment about the trailing edge (positive nose-down) is

$$M_{te} = \rho U\int_{le}^{te} \gamma(x,t)x\,\mathrm{d}x - \frac{1}{2}\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{le}^{te} \gamma(x,t)x^2\,\mathrm{d}x \tag{9}$$

These formulas can be obtained by integration of $\gamma_{\text{eff}}(x, t)$, as given in Eq. (7) (involving, in each case, an integration by parts). Alternatively, they are obtained from the rates of change of the vertical impulse and moment of impulse of the vortex sheets.[4,7,8] Equation (8) was given in Refs. 8 and 9; so far as we know, Eq. (9) has not been published.

How can these formulas be correct if the pressure at the airfoil surface is in error because of thick separated layers, eddies, etc.? The obvious answer is that only the detailed distribution of load is in error, and not the total lift and moment.

Conclusions

Besides the two categories of airfoil flows discussed here, viz., those with "enforced" separation at the trailing edge and those with "natural" separation at upstream locations, there is obviously a most important category in which there is "natural" separation on one surface and "enforced" separation at the trailing edge on the other. But this does not require additional discussion. In principle, all of these categories are the same, for the boundary-layer calculations mentioned repeatedly here always will determine the separation location, be it either at the trailing edge or upstream of it.

Howarth satisfied the circulation criterion, Eq. (3) with right-hand side equal to zero, by means of an iterative procedure, trying successive values of Γ until agreement was obtained. This also will be the case for unsteady flow, but a bit more complicated. Williams and Johnson, as already has been mentioned, are working with this kind of iterative calculation for sinusoidal airfoil motions. The content of this paper has been discussed with them, and the results of their investigations are awaited with great interest. They should cast important light on such subjects as lift-hysteresis and its dependence upon boundary-layer properties.

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